MATHEMATICS

# A CORRESPONDENCE BETWEEN TWO SETS OF TREES

BY

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*Synopsis*

A *planted, edge chromatic tree on $k$ colors* $(V, E, v; f)$ is a tree with *vertex set* $V$, *edge set* $E$, a *root* $v \in V$ with degree 1, and a mapping $f$ sending $E$ into a $k$-set of colors so that adjacent edges are mapped to different colors. Riordan [5] found formulas for $t(n, k)$, the number of classes of isomorphic, planted, edge chromatic trees on $k+1$ colors having $n+1$ vertices, and every tree bearing the same color on its stem; however, Riordan overlooked the elegant formula $t(n, k) = \binom{kn}{n} / (kn - n + 1)$ which was recently pointed out by Carlitz [1]. It was shown in [4] that $p(n, k) = \binom{kn}{n} / (kn - n + 1)$, where $p(n, k)$ denotes the number of classes of isomorphic, $(k+1)$-valent, planted plane trees having $kn+2$ vertices. Since $t(n, k) = p(n, k)$, the question of constructing a "natural" one to one correspondence between the classes of isomorphic trees involved arises. Such a correspondence is provided in this note; also, precise definitions of the trees under discussion are given in terms of binary relations on finite sets which we hope will replace the topological definitions used formerly.

*Definitions and notation*

A *tree* is a connected linear graph $(V, E)$ having no cycles; $V$ is the *vertex set*, $E$ is the *edge set*, and $E$ is a set of 2-sets of $V$. (If $(V, E)$ is a tree, then $|V| = |E| + 1$.) A *rooted tree* $(V, E, v)$ is a tree $(V, E)$ with a distinguished vertex $v \in V$ called the *root*; a rooted tree is a *planted tree* if the degree of the root is 1. Let $K$ denote a $k$-set, then an *edge chromatic tree* $(V, E; f)$ *on* $K$ (alternatively, an *edge chromatic tree on $k$ colors*) is a tree $(V, E)$ and a mapping $f$ sending $E$ into $K$ such that adjacent edges are mapped to distinct elements of $K$. Let $T(V, K)$ denote the set of all planted, edge chromatic trees on $K$ having vertex set $V$, and every tree in $T(V, K)$ has the same color $c \in K$ assigned to the rooted edge. Two elements $(V, E_1, v_1; f_1)$, $(V, E_2, v_2; f_2) \in T(V, K)$ are *isomorphic* if there exists a permutation $\varphi$ of $V$ such that (i) $v_2 = \varphi(v_1)$, (ii) $E_2 = \{\{\varphi x, \varphi y\} : \{x, y\} \in E_1\}$, and (iii) $f_1\{x, y\} = f_2\{\varphi x, \varphi y\}$. Isomorphism is an equivalence relation on $T(V, K)$; let $T^*(V, K)$ denote the set of equivalence classes defined in

$T(V, K)$ by isomorphism. Suppose $|V| = n+1$, $|K| = k+1$, and let $t(n, k) := |T^*(V, K)|$. The elements of $T^*(V, K)$ are equivalent to what RIORDAN [5] called chromatic planted trees with $k+1$ line colors, $n$ lines, and a given color on the stem. CARLITZ [1] used Riordan's equation involving the generating function of $\{t(1, k), t(2, k), \ldots\}$ to deduce that

$$t(n, k) = \binom{kn}{n} / (kn - n + 1). \tag{1}$$

Suppose $(V, E, v)$ is a rooted tree and let $\varrho(x)$ denote the length of the path from $v$ to $x$ for each $x \in V$; in particular, $\varrho(v) = 0$. A *planted plane tree* $(V, E, v, R)$ is a planted tree $(V, E, v)$ with a linear order relation $R$ on $V$ possessing two properties: (i) If $x, y \in V$ and $\varrho(x) < \varrho(y)$, then $(x, y) \in R$. (ii) If $\{r, s\}, \{x, y\} \in E$, $\varrho(r) = \varrho(x) = \varrho(s) - 1 = \varrho(y) - 1$, and $(r, x) \in R$, then $(s, y) \in R$. Let $P(V)$ denote the set of planted plane trees with vertex set $V$. Planted plane trees as we have defined them are equivalent to the planted plane trees defined by HARARY, PRINS, and TUTTE [3]; our definition is in terms of binary relations on finite sets while the latter definition is given in topological terms.

There are nice diagrams corresponding to planted plane trees; the vertices of the tree can be arranged in consecutive levels 0, 1, ... so that $x \in V$ is in level $\varrho(x)$, and then the vertices in each level can be put in a linear order from left to right according to the order relation $R$. The conditions on $R$ imply that the edges of the tree will occur in the diagram between consecutive levels, and if the edges of the tree are indicated by straight lines in the diagram, then no lines intersect in the diagram. Thus, planted plane trees are equivalent to topological trees that have been embedded in the plane. The diagrams of the elements of $P\{a, b, c, d, e, f\}$ corresponding to a given planted tree are shown in figure 1.

Two elements $(V, E_1, v_1, R_1)$, $(V, E_2, v_2, R_2) \in P(V)$ are *isomorphic* if there exists a permutation $\varphi$ of $V$ such that (i) $v_2 = \varphi(v_1)$, (ii) $E_2 = \{\{\varphi x, \varphi y\} : \{x, y\} \in E_1\}$, and (iii) $(x, y) \in R_1$ if and only if $(\varphi x, \varphi y) \in R_2$. Isomorphism is an equivalence relation on $P(V)$; hence, let $P^*(V)$ denote the classes of isomorphic, planted plane trees with vertex set $V$. (Elements of $P^*(V)$ are equivalent to what HARARY, PRINS and TUTTE [3] called map-isomorphic, planted plane trees.)

A tree is *k-valent* if every vertex of the tree has degree 1 or $k$. Let $P(V, k)$

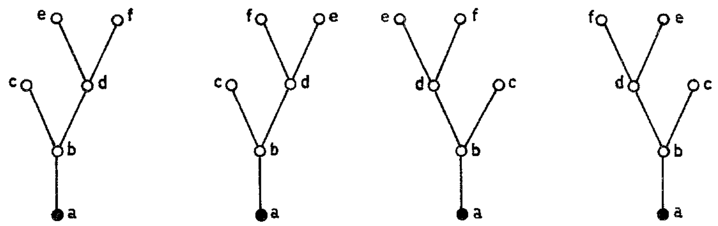


Fig. 1. Some planted plane trees.

denote the set of all $(k+1)$-valent, planted plane trees with vertex set $V$, and let $P^*(V, k)$ denote the set of classes of isomorphic planted plane trees in $P(V, k)$. For a fixed $k$ it is easy to prove by induction on $n$ that $P(V, k) \neq \phi$ implies $|V| = kn+2$ for some $n$. Suppose $|V| = kn+2$, and let $p(n, k) := |P^*(V, k)|$. It was shown in [4] that

$$p(n, k) = \binom{kn}{n} / (kn-n+1). \tag{2}$$

The diagrams corresponding to the $p(4, 2) = \binom{8}{4}/5 = 14$ elements of $P^*(V, 2)$ with $|V| = 2(4)+2 = 10$ are shown in figure 2.

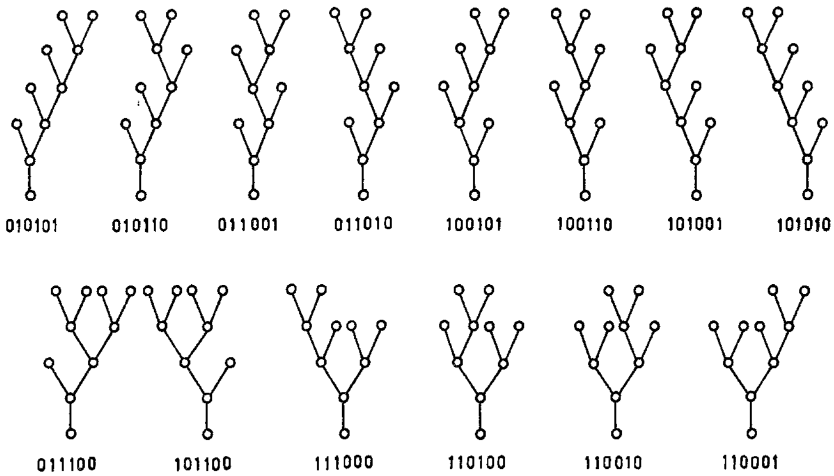


Fig. 2. Diagrams representing the elements of $P^*(V, 2)$, $|V| = 10$, with the corresponding code words.

Let $B(n, k)$ denote the set of all binary sequences $(b_1, \ldots, b_{nk-k})$ containing exactly $n-1$ units such that $b_1 + \ldots + b_{jk} \geqslant j$ for $j = 1, \ldots, n-1$; also, let $b(n, k) := |B(n, k)|$. We observed in [4] that there is a one to one correspondence $\zeta$ between the elements of $B(n, k)$ and $P^*(V, k)$ when $|V| = kn+2$; thus, $b(n, k) = p(n, k)$.

Now we are going to construct a one to one correspondence $\xi$ between the elements of $B(n, k)$ and the elements of $T^*(V, K)$ with $|V| = n+1$ and $|K| = k+1$. Once this has been accomplished, $\zeta$ and $\xi$ can be combined to obtain a one to one correspondence between the elements of $P^*(U, k)$ with $|U| = kn+2$, and the elements of $T^*(V, K)$ with $|V| = n+1$ and $|K| = k+1$.

*The correspondence $\xi$*

Let $K$ denote a $(k+1)$-set; we will require a linear order relation on $K$, so we assume $K = \{1, \ldots, k+1\}$ in the first place and use the natural ordering of the integers. Let $V$ denote an $(n+1)$-set, and consider an element $T := (V, E, v; f) \in T(V, K)$. Now we define a linear order relation $\leqslant$ on $V$ in terms of $E$, $v$, and $f$; however, we plan to use the symbol "$\leqslant$"

ambiguously and depend on the context in which it is used to make its meaning clear. (Recall that $\varrho(x)$ denotes the length of a path from $x \in V$ to the root $v$ in $T$.) First, define $x<y$ in $T$ if $\varrho(x)<\varrho(y)$; next, vertices belonging to the same level in $T$ can be linearly ordered by induction on the levels of the vertices by using the following rule: If $\{r, s\}$, $\{x, y\} \in E$, $\varrho(r)=\varrho(x)=\varrho(s)-1=\varrho(y)-1$, then define $s \leqslant y$ in $T$ if and only if $f\{r, s\} \leqslant f\{x, y\}$. (It is easy to check that if $\varphi$ is an isomorphism sending $T_1$ onto $T_2$ where $T_1, T_2 \in T(V, K)$, then $x \leqslant y$ in $T_1$ if and only if $\varphi x \leqslant \varphi y$ in $T_2$; just use the fact that $\varrho(x)=\varrho(\varphi x)$, and $f_1\{x, y\}=f_2\{\varphi x, \varphi y\}$.)

Now we assign a binary sequence of length $nk-k$ to each element of $T(V, K)$. Suppose $T:=(V, E, v; f) \in T(V, K)$, and suppose $v=v_0<v_1<\ldots<v_n$ is the linear order on $V$ in $T$ defined in the last paragraph, then $\xi(T)=(b_1, \ldots, b_{nk-n})$ is defined as follows: For $j=1, \ldots, n$ let $a_j$ denote the color assigned to the edge leading from $v_j$ to the root $v$; also, suppose the elements of $K\setminus\{a_j\}$ are $c_1<\ldots<c_k$; now, define $b_{kj+i}=1$ if some edge joined to $v_j$ is assigned the color $c_i$, and define $b_{kj+i}=0$ otherwise for $i=1, \ldots, k$. It is a routine matter to check that $\xi(T_1)=\xi(T_2)$ if and only if $T_1$ and $T_2$ are isomorphic. Thus, indulging in a slight abuse of notation, $\xi$ can be considered a mapping sending $T^*(V, K)$ into $B(n, k)$. To see that $\xi(T) \in B(n, k)$ for every $T \in T(V, K)$, note that $b_1+\ldots+b_{kj}$ is the sum of the degrees of vertices $v_1, \ldots, v_j$, and $T$ with its rooted edge deleted is a tree. Also, $T$ with its rooted edge deleted is a tree with $n$ vertices, so $(b_1, \ldots, b_{kn-k})$ has exactly $n-1$ units, each unit corresponding to an edge in the tree. Conversely, given $(b_1, \ldots, b_{kn-k}) \in B(n, k)$, a representative element of a class in $T^*(V, K)$ can be constructed. The diagrams corresponding to the elements of $T^*(V, K)$ with $|V|=n+1=5$ and $|K|=k+1=3$ are shown in figure 3; in the diagrams an edge assigned $i \in K$ has been indicated with $i$ parallel lines. The binary sequence of length $(4-1)\times(3-1)=6$ assigned to each tree by $\xi$ is also indicated in the figure.

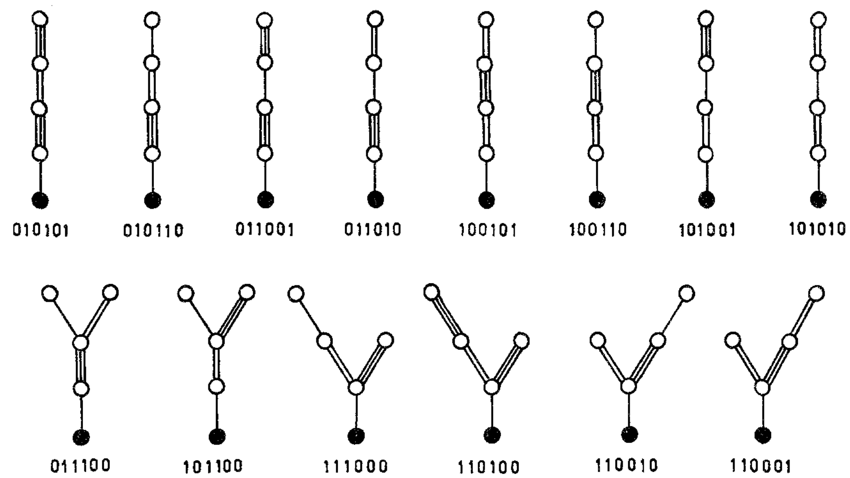


Fig. 3. Diagrams representing the elements of $T^*(V, K)$, $|V|=5$, $|K|=3$ with the corresponding code words.

N. G. de Bruijn has observed that the correspondence $\xi$ is the same as the one suggested by figure 4. Suppose $X \in P^*(V, k)$, $|V| = kn+2$, and select a representative element $T \in X$. There is a unique subtree $T'$ of $T$ which has as its vertex set the root and all vertices of degree $k+1$ in $T$; since $T$ has exactly $n$ vertices of degree $k+1$, $T'$ has exactly $n+1$

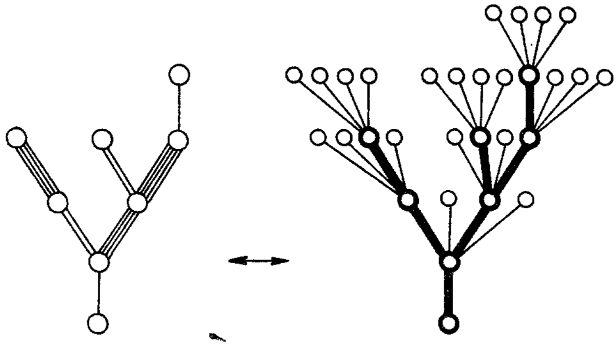

Fig. 4.

vertices. We are going to color the edges of $T$ so that $T'$ without the linear order relation on its vertices becomes an element of $T^*(V', k)$, $|V'| = n+1$. Let $K = \{1, \ldots, k+1\}$ denote a set of colors, and assign 1 to the rooted edge of $T$. Begin with the edges adjacent to the rooted edge of $T$: Suppose the edge below a vertex $x$ of degree $k+1$ (the edge going toward the root from $x$) has been colored $i \in K$, and suppose $c_1 < \ldots < c_k$ are the colors in $K \backslash \{i\}$, then the $k$ edges above and adjacent to $x$ are colored $c_1, \ldots, c_k$ from left to right. This coloring of $T$ is also a coloring of $T'$, and $T' \in T(V', K)$. It is easy to check that this mapping of $P(V, k)$ into $T(V', K)$ defines a one to one correspondence between the elements of $P^*(V, k)$ and $T^*(V', k)$ which is equivalent to $\xi$.

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